ational method automatically adjusts the amplitudes of surface-wave modes in the appropriate manner to construct the system modes, but the various integrals in Section II will diverge if radiation modes are used, so that some modification of the present method would be necessary to treat problems of this type.

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Single-Mode Pulse Dispersion in Optical Waveguides

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Abstract—**The limitations of a widely used method for analyzing pulse distortion in a single-mode waveguiding structure are derived. The results are applied to propagation in optical waveguides, and for cases where material dispersion is dominated by a broad resonance line, pulse attenuation is found to be much more serious than the broadening of the pulse. In extremely low-loss regions, however, other effects may cause the reverse to be true.**

I. INTRODUCTION

With the recent development of extremely low-loss optical waveguides [1] making feasible long-distance transmission via this medium, there has been increased interest in determining the pulse characteristics of such devices [2], [3]. These characteristics are determined by the nonlinearity of the $\beta - \omega$ characteristics of individual modes, and, in multimode guides, by the differences in group velocity between different modes. In this paper we address ourselves to the first of these causes, referring the reader to [3] for a discussion of the second.

We shall obtain a more precise formulation for the region of validity of a widely used technique [2], [4]–[6] for analyzing, approximately, the distortion of a single pulse, and apply these results to the study of such pulses in optical waveguides.

Manuscript received March 19, 1975; revised June 5, 1975. This work was supported by the Air Force Office of Scientific Research under Grant AFOSR-72-2417.

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II. PROPAGATION OF A GAUSSIAN PULSE

We seek to analyze the behavior of a pulse of Gaussian envelope

$$f(t) = (a\sqrt{\pi})^{-1/2} \exp(-t^2/2a^2) \exp(j\omega_0 t) \qquad (1)$$

with center frequency ω_0 and width a which has been normalized to unit strength

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = 1$$

as it propagates along an arbitrary transmission channel of transfer function $S(\omega) = \exp[-j\beta(\omega)L]$. Here L is the length of a section of the channel between the input ($z = 0$) and the output, and $\beta(\omega) = h(\omega) - \frac{1}{2}j\alpha(\omega)$ is the frequency-dependent propagation constant of the channel split into phase constant and (power) attenuation constant. The output signal is then represented by the usual Fourier-transform method

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) S(\omega) \exp(j\omega t) \, d\omega \qquad (2)$$

where $F(\omega)$ is the transform of $f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt = (2a\sqrt{\pi})^{1/2} \exp[-\tfrac{1}{2}a^2(\omega - \omega_0)^2].$$

We shall assume throughout the paper that $f(t)$ is "quasi-monochromatic"; i.e., that $a\omega_0 \gg 1$ so that $F(\omega)$ is significantly different from zero only in a very narrow bandwidth about ω_0.

Commonly [2], [4]–[6], this type of problem has been analyzed by expressing $\beta(\omega)$ as a Taylor series about ω_0

$$\beta(\omega)L = \{\beta_0 + (\omega - \omega_0)\beta_0' + \tfrac{1}{2}(\omega - \omega_0)^2\beta_0'' + \tfrac{1}{6}(\omega - \omega_0)^3\beta_0''' + \cdots\}L \quad (3)$$

where primes denote differentiation with respect to ω, and the subscript zero indicates evaluation at ω_0. If the series (3) is truncated after the third term, it is a straightforward matter to evaluate (2) in closed form [2] to obtain

$$q(t) = r(t - L/v_{g0}) \exp[-\tfrac{1}{2}\alpha_0 L + j\omega_0(t - L/v_{p0})] \quad (4)$$

where $v_p = \omega/h(\omega)$ is the phase velocity and $v_g = 1/h'(\omega)$ is the group velocity. The envelope of $r(t)$ is given by

$$| r(t) |^2 = (b\sqrt{\pi})^{-1} \frac{a}{a_0} \exp\left\{-\left(t + \frac{1}{2}\frac{\alpha_0' h_0'' L^2}{a_0^2}\right)^2 \Big/ b^2 + \frac{1}{4}\frac{\alpha_0'^2 L^2}{b^2}\left[1 + \left(\frac{h_0'' L}{a_0^2}\right)^2\right]\right\} \quad (5)$$

where

$$b = [a_0^2 + (h_0'' L/a_0)^2]^{1/2} \quad a_0 = [a^2 + \tfrac{1}{2}\alpha_0'' L]^{1/2}. \quad (6)$$

Thus, outside of amplitude factors, phase delays, and group delays, this analysis predicts that the signal remains Gaussian but is broadened from the width a to the width b.

The usual justification for the foregoing method is that the spectrum $F(\omega)$ is negligible away from ω_0, and any error caused by truncation of the Taylor series (3) will therefore not be significant. However, for sufficiently large L all spectral components become important, and just when (4)–(6) cease to be a valid approximation to the output is unclear. Some improvement might be expected if, say, the fourth term of (3) were retained, as done in [7]; however, because (3) has a finite radius of convergence, retention of additional terms of (3) will not, in general, improve the region of applicability of the solution obtained in this way [5]. Furthermore, quite apart from the type of signal $f(t)$ we are considering, the three-term approximation to $\beta(\omega)$ does not represent a causal dispersion relation [8]. Finally, if only three terms of (3) are used, we must have $a^2 + \frac{1}{2}\alpha_0'' L > 0$ for the integral (2) to converge, and it is not clear how essential this condition is.

What is needed, then, is a way to "pull out" the Taylor series from under the integral sign somehow so that the error incurred by its truncation may be more easily estimated. One way to do this [6], [9]–[11] is to apply the method of saddle points [12] to (2), where we may take $a^2\omega_0^2$ as the large parameter by virtue of the narrow assumption on $f(t)$. We thus rewrite (2) as

$$q(t) = \frac{(2a\sqrt{\pi})^{1/2}}{2\pi} \int_{-\infty}^{\infty} \exp[a^2\omega_0^2\phi(\omega)]\, d\omega \quad (7)$$

where

$$\phi(\omega) = -\frac{1}{2}\left(\frac{\omega}{\omega_0} - 1\right)^2 + j\left[\frac{\omega t - \beta(\omega)L}{a^2\omega_0^2}\right]. \quad (8)$$

Any saddle point ω_s must satisfy

$$0 = \phi'(\omega_s) = -\frac{1}{\omega_0}\left(\frac{\omega_s}{\omega_0} - 1\right) + j\left[\frac{t - \beta'(\omega_s)L}{a^2\omega_0^2}\right] \quad (9a)$$

or equivalently

$$\omega_s = \omega_0 + j\,\frac{t - \beta'(\omega_s)L}{a^2}. \quad (9b)$$

Assuming for the moment that the path of integration may be deformed from the real axis into a steepest descent path (SDP) passing through the saddle point ω_s in the complex ω plane, and also that only one saddle point contributes to the evaluation of (7) (we shall examine these points more closely), $q(t)$ may be represented by the first term of the saddle-point asymptotic series

$$q(t) \sim [(a\sqrt{\pi})^{1/2}\omega_0]^{-1}[-\phi''(\omega_s)]^{-1/2} \exp[a^2\omega_0^2\phi(\omega_s)]. \quad (10)$$

The branch of $[-\phi''(\omega_s)]^{-1/2}$ is chosen so that

$$\arg[-\phi''(\omega_s)]^{1/2}$$

is equal to the slope of the SDP at ω_s with respect to the positive real axis.

At this point there is no integration to perform and there remains only to calculate $\phi(\omega_s)$ and $\phi''(\omega_s)$ (since ω_s is, in general, complex and we presumably have information about $\beta(\omega)$ only on the real axis of the complex ω plane, and only at certain points) in terms of quantities evaluated at ω_0. Since now ω_s is defined implicitly and functions evaluated at ω_s are required, the simple Taylor series (3) no longer serves our purposes; we make use of a series expansion due to Levi-Cività [13] (details of the derivation for the complex case with remainder term must be assembled from [14] and [15] since the infinite series only is given without proof in [13]). Briefly, given functions f and ψ analytic over suitable regions and a variable y defined implicitly as a function of another variable x by $y = x + \psi(y)$, we may expand $f(y)$ as

$$f(y) = f(x) + \sum_{m=1}^{n-1} \frac{1}{m!}[\psi(x)]^m \cdot \left\{\frac{1}{1 - \psi'(x)}\frac{d}{dx}\right\}^m f(x) + R_n$$

$$R_n = \frac{\lambda[\psi(x)]^n}{n!}\left[\left\{\frac{1}{1 - \psi'(w)}\frac{d}{dw}\right\}^n f(w)\right]$$

and x and w are related by $w - \psi(w) = x - \theta\psi(x)$. Here θ is some real number with $0 \leq \theta \leq 1$, and λ is some complex number with $|\lambda| \leq 1$.

If we take $f(y) = \phi(\omega_s)$, $y = \omega_s$, $x = \omega_0$, $\psi(y) = j[t - \beta'(\omega_s)L]/a^2$, and $n = 4$, we have

$$\phi(\omega_s) = j(\omega_0 t - \beta_0 L)(a^2\omega_0^2)^{-1} - (t - \beta_0' L)^2 \cdot [2a^2\omega_0^2(a^2 + j\beta_0'' L)]^{-1} - \beta_0''' L(t - \beta_0' L)^3 \cdot [6a^2\omega_0^2(a^2 + j\beta_0'' L)^3]^{-1} + 0[(t_0 - \beta_0' L)^4]. \quad (11)$$

Similarly, if we use $f(y) = \phi''(\omega_s)$ and $n = 1$, we get

$$\phi''(\omega_s) = -[1 + j\beta_0'' L/a^2]/\omega_0^2 + R_1 \quad (12)$$

where

$$R_1 = \frac{\lambda}{a^2\omega_0^2} \frac{\beta'''(\Omega)L(t - \beta_0' L)}{a^2 + j\beta''(\Omega)L}.$$

The point Ω satisfies

$$\Omega = \omega_0 - j\theta(t - \beta_0' L)/a^2 + j[t - \beta'(\Omega)L]/a^2$$

for some constant θ on $[0,1]$. λ is a complex number of magnitude not greater than one, as indicated previously.

The insertion of (11) and (12) into (10), neglecting R_1 and the third and fourth terms on the right-hand side of (11), yields precisely (4)–(6). If we may, for purposes of error estimation, approximate functions evaluated at Ω by the corresponding values at ω_0 (this is reasonable if $|\omega_s - \omega_0|$ is not too large, an assumption consistent with the upper bounds on L found later in this paper), then this neglect requires that we satisfy the conditions

$$\frac{|t - \beta_0' L||\beta_0''' L|}{|a^2 + j\beta_0'' L|^2} \ll 1 \quad (13)$$

and a similar condition involving $|\beta^{[iv]} L|$ which can be shown to be satisfied for t and L of interest as long as (13) is.

III. INTERPRETATION AND LIMITATIONS OF THE RESULTS

The condition (13) for the validity of (4)–(6) is not independent of t, but holds only for sufficiently small $|t - \beta_0' L|/|a^2 + j\beta_0'' L|^{1/2}$. Thus the Gaussian character of the output pulse is preserved only if we follow closely enough the crest of the pulse envelope, whereas the tails of the pulse may be distorted in an unknown way. Equations (4)–(6) will still be meaningful if they retain validity for the most significant portion of the envelope; i.e.

$$|t - \beta_0' L|/|a^2 + j\beta_0'' L|^{1/2} \lesssim 5.$$

Therefore, an equivalent reformulation of (13) is seen to be

$$|\beta_0''' L|/5|a^2 + j\beta_0'' L|^{3/2} \ll 1. \quad (14)$$

(Notice that β_0''' does not necessarily have to be small compared to $\beta_0'' a$.) In fact we do not need to concern ourselves at all with the pulse tails, since the Gaussian is nonzero for all t and therefore does not represent a causal signal in the sense of information transmission. It is possible to truncate the signal at points of arbitrarily small amplitude on the tails such that the transient pre- and postcursors resulting therefrom will have arbitrarily small amplitude compared with that of the Gaussian [11]. The details of dispersive propagation of signals with such abrupt turn-on and turn-off points have been described elsewhere [16].

Thus the additional time-displacement term $\alpha_0' h_0'' L^2/2a_0^2$ in (5) may not be explained in terms of the noncausality of the signal [17]. Instead, it must be realized that the presence of frequency-dependent absorption results in a differential attenuation between different frequency components of the pulse which results in the previously mentioned displacement of the pulse maximum. One implication of this is that, in absorbing transmission channels, the classical group velocity no longer gives the velocity of energy propagation, a point discussed further in [5], [6], [9], [17], and [18].

Another condition of validity for (4)–(6) not reflected in (13) is that the contour deformation from the real axis to the SDP in (7) be permissible, and that only a single saddle point in the neighborhood of ω_0 contribute to the asymptotic evaluation of the integral. In the first place, any singularities of $\beta(\omega)$ encountered in the deformation must be such that their contribution to the evaluation of (7) is small compared to that of the saddle point as a result of being damped out by the Gaussian envelope. If this were not the case, ω_s would be so close to the singularity that the lower order derivatives of $\beta(\omega)$ would be expected to become large (as discussed at the end of the previous section) and (4)–(6) lose validity anyway.

For $L > 0$ but sufficiently small, the inverse function theorem [19] assures us that $\omega_s(L)$ is unique, at least in the neighborhood of ω_0, if $1 + j\beta''(\omega)L/a^2 \neq 0$ in a suitable neighborhood of ω_s. It can be shown that condition (13) is sufficient to assure us this uniqueness. Unless such a zero occurs near ω_s, the Levi-Cività series unambiguously gives an expression for $\omega_s(L)$, and any other saddle points are assumed either not to be encountered on the SDP deformed from the real axis, or to contribute an exponentially small term, due again to the damping of the Gaussian. Now from well-known properties of materials and waveguides, we have it that for sufficiently high frequency $\beta(\omega) \sim \omega/c$ [20], [21]. This behavior is not enough to essentially alter that of $\phi(\omega)$ which is dominated by the Gaussian. In the general case, the regions where $\mathrm{Re}[\phi(\omega) - \phi(\omega_s)] < 0$ can be described asymptotically and are as shown in Fig. 1. The contour deformation is readily shown to be a permissible one.

It should be noted here that the condition $a^2 + \frac{1}{2}\alpha_0'' L > 0$, which is required to assure the convergence of the inversion integral when (4)–(6) are derived in the conventional manner [5], is, as such, not necessary here. The condition that is required is that $1 + j\beta''(\omega)L/a^2$ have no zeros in a sufficiently large neighborhood of ω_s; more precisely, that such zeros ω_B, if they exist, should be such that $a^2\omega_0^2|\tau_B|^2 \gg 1$ so as not to affect the asymptotic

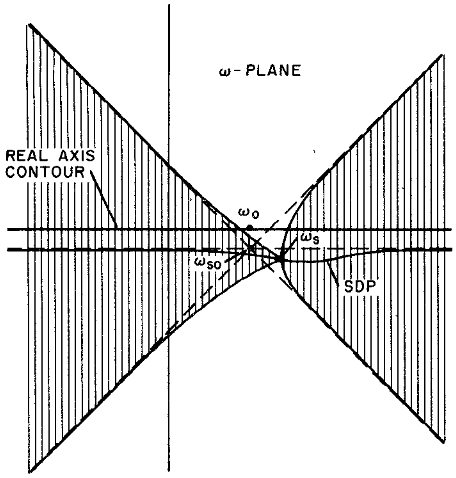

Fig. 1. Contours of integration. Shaded area indicates Re $[\phi(\omega) - \phi(\omega_s)] < 0$, where SDP may be joined to the real axis contour at infinity. Here $\omega_{s0} = \omega_0 + j(t - L/c)/a^2$.

evaluation of (7). A rough error estimate similar to the ones given previously shows that (again, if we are not near any singularity of $\beta(\omega)$, a self-consistent hypothesis) condition (13) is sufficient to achieve this separation of ω_B and ω_s.

In the Appendix, a specific functional form of $\beta(\omega)$ is examined in light of the discussion of this section.

IV. DISPERSION LENGTH CRITERIA

Using a parameter $A = L/a^2$, we define and examine a function $G(A)$ which measures the relative broadening of the pulse

$$G(A) = a^2/b^2 = \frac{1 + \frac{1}{2}\alpha_0'' A}{1 + \alpha_0'' A + |\beta_0''|^2 A^2} \qquad (15)$$

valid under the conditions outlined in the previous section. If we consider a train of such pulses at a rate of N pulses/s (with a corresponding duty cycle $\Delta = Na$) there is a certain amount of overlap between adjacent pulses. For a given value of A, the relative power level of one of the pulses (compared to the power level at the peak of the pulse) midway between adjacent pulses is given by

$$\delta = \exp[-1/(2Nb)^2] = \exp[-G(A)/4\Delta^2]. \qquad (16)$$

This overlap level has its minimum value δ_0 at $A = 0$.

In a practical optical communication system, acceptable values of δ may be restricted by the capabilities of system components other than the transmission channel (pulse generators, detectors, and so on). It therefore seems most natural to assume that system considerations result in a maximum allowable overlap level of $\delta_{\max}$, rather than attempting to maximize the pulse rate as if the channel were in isolation, as done in [2]. A maximum usable propagation length or dispersion length is then given by $L_0 = a^2 A_0$, where A_0 is the positive root of

$$G(A_0) = G_0 = -4\Delta^2 \ln \delta_{\max}. \qquad (17)$$

It is readily seen that $\delta_0 < \delta_{\max}$ is required for a positive root to exist.

In two particular cases, the solution to (17) takes rather simple forms. If $|\frac{1}{2}\alpha_0''| \ll |h_0''|$ (such as in low-attenuation channels)

$$A_0 \simeq |h_0''|^{-1}[(1 - G_0)/G_0]^{1/2} \qquad (18)$$

gives explicitly the dispersion length L_0 in terms of $\delta_{\max}$ according to (17). Furthermore, if one assigns $G_0 = \frac{1}{2}$ (which is roughly a 40-percent broadening), (17) gives exactly

$$A_0 = |\beta_0''|^{-1}. \qquad (19)$$

V. DISPERSION IN OPTICAL WAVEGUIDES

For single-mode propagation over real waveguides, $h''(\omega)$ and $\alpha''(\omega)$ depend both on the geometry of the waveguide and on the dispersive properties of the materials used. Although, in general, the two effects may be of comparable magnitude, under many conditions the material effect is dominant [22] (in particular this is true for dispersion-optimized fibers [23]). There even exist structures in which the two effects can effectively cancel one another, thus allowing extremely low-dispersion operation [24]. Although in general the two effects are not simply additive [23], they may be examined separately to obtain order-of-magnitude estimates for guide information rates.

Waveguide geometry effects have been studied for many cases [2], [22]–[25]. A number of lossless and lossy low-permittivity contrast (or weakly guiding) inhomogeneous slab waveguides were studied in [25], and even for guides with fairly high (500-dB/km) frequency-independent loss, $|\frac{1}{2}\alpha''|$ was found to be at least an order of magnitude smaller than $|h''|$ in all cases, allowing the use of (18). The largest dispersion found for any of the guides analyzed was $2|h_0''| \simeq 10^{-24}$ s^2/m, so that a pulse of width $a = 3 \times 10^{-11}$ s is broadened only 10 percent over a distance $L = 1$ km (corresponding to a maximum pulse rate of $N = 30 \times 10^9$ pulses/s for this broadening criterion). Fig. 2 shows a representative slab waveguide and plot of a normalized version of $|h''|$ versus frequency taken from [25].

In some situations, the material dispersion can be well represented by a fairly broad resonance-type dispersion law such as in the Appendix (the OH-radical absorption due to the presence of water molecules in glass fibers is a good example of this [26]). Away from the influence of such resonances, however (especially in the extremely low-attenuation regime of fiber materials such as silica), other more complicated considerations will dominate the behavior of $\beta(\omega)$ [27]–[29]. Since it is shown in the Appendix that, except for a region near the resonance peak where the present theory does not apply, attenuation is a much more serious problem than single-mode dispersion, and also since experimental measurements on low-loss fibers indicate that significant pulse broadening is possible with minimal attenuation [30], such simple models are evidently inadequate to accurately describe the behavior of useful long-distance waveguide materials.

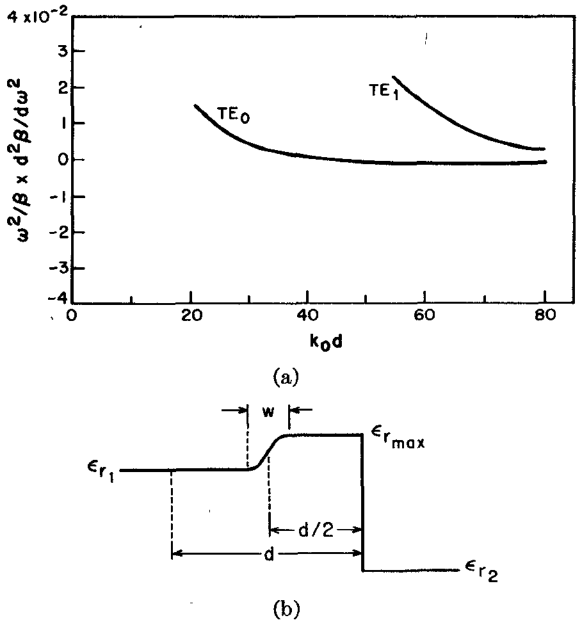

Fig. 2. (a) Normalized TE-mode-broadening parameter ($d^2\beta/d\omega^2$) for asymmetric slab waveguide with $\epsilon_{r1} = 1.50$, $\epsilon_{r2} = 1.00$, $\epsilon_{r_{\max}} = 1.53$, $d = 4 \times 10^{-6}$ m, $w/d = 0.2$. (b) Permittivity profile.

VI. CONCLUSION

An analysis of single-mode pulse distortion has been carried out using a method which enables the region of applicability for a widely used approximation to be more precisely specified according to (13). The model dielectric dispersion law used in [5] is examined. The validity criterion contained therein is found to be in error, and the correct criteria in light of the present derivation are given. Applying the results to the properties of optical waveguides, it is found that unless low-loss long-distance guides are considered, material dispersion considerations limit operation primarily by signal attenuation rather than by pulse broadening. This is true in particular for the relatively lossy materials tolerable in integrated optical devices.

APPENDIX

In [5], a study was made of the propagation of a Gaussian light pulse through a medium whose wavenumber $\beta(\omega)$ obeyed the law

$$\beta(\omega) = \frac{\omega n_\infty}{c} - \frac{\omega_p \omega_c}{c(\omega - \omega_c - j\gamma)} \tag{20}$$

where n_∞, ω_c, ω_p, and γ are all real adjustable constants and c is the speed of light. The Gaussian pulse of width a described by (1) is then propagated and broadened as described in Section II. It was assumed in [5] that the resonance of the medium was broad compared to the pulse bandwidth but still narrow compared with the carrier frequency, and that the resonance peak dominated the attenuation characteristic $\alpha(\omega)$ while only weakly affecting $h(\omega)$. In other words

$$\omega_0 a \gg \gamma a \gg 1$$

$$\omega_p/\gamma \ll n_\infty. \tag{21}$$

For some range of t and L, then, equations (4)–(6) apply, with

$$h_0'' = -2\frac{\omega_p\omega_c}{c}\frac{(\omega_0 - \omega_c)[(\omega_0 - \omega_c)^2 - 3\gamma^2]}{[(\omega_0 - \omega_c)^2 + \gamma^2]^3}$$

$$\tfrac{1}{2}\alpha_0'' = 2\frac{\omega_p\omega_c}{c}\frac{\gamma[3(\omega_0 - \omega_c)^2 - \gamma^2]}{[(\omega_0 - \omega_c)^2 + \gamma^2]^3}$$

with the region of validity given by (14) with

$$|\beta_0'''| = \frac{6\omega_p\omega_c}{c}[(\omega_0 - \omega_c)^2 + \gamma^2]^{-2}.$$

Direct substitution into (14) gives a rather messy criterion for L; however, sufficient bounds on L may be obtained by considering two cases.

Case 1: $L > -\alpha_0''a^2/|\beta_0''|^2$. This implies $a^2 < |a^2 + j\beta_0''L|$ so that we may replace $|a^2 + j\beta_0''L|$ by a^2 in (14) to obtain a sufficient condition

$$L \ll \frac{ca^3}{\omega_p\omega_c}[(\omega_0 - \omega_c)^2 + \gamma^2]^2. \tag{22a}$$

Case 2: $L < -\alpha_0''a^2/|\beta_0''|^2$. Here we may multiply the left side of (14) by $a|a^2 + j\beta_0''L|^{-1/2}$ to obtain the sufficient condition

$$L \ll \frac{c}{\omega_p\omega_c}\frac{[(\omega_0 - \omega_c)^2 + \gamma^2]^2}{a}[a^4 + \alpha_0''La^2 + |\beta_0''L|^2]. \tag{22b}$$

The criterion given in [5, eq. (30)] for the validity of (4)–(6) can be written in the present notation as

$$L \ll \frac{ca^3}{\omega_p\omega_c}\frac{a^2\gamma^2[(\omega_0 - \omega_c)^2 + \gamma^2]}{a^2 + \frac{1}{2}\alpha_0''L}. \tag{23}$$

The presence of the term $a_0^2 = a^2 + \frac{1}{2}\alpha_0''L$ in the denominator indicates either a typographical error or an analytical one, since allowing $a_0^2 \to 0$ implies validity for all L which cannot be true as can be easily seen if we suppose we are operating on the resonance peak ($\omega_0 = \omega_c$). Since the derivation of (23) was not presented in [5] it is not possible to directly compare it with the criteria (22); however, this does point up the need to use (4)–(6) with caution.

Finally, consider a point of operation moderately removed from resonance ($|\omega_0 - \omega_c|^2 > \gamma^2/3$). If (4)–(6) are valid, then a 40-percent pulse broadening occurs at

$$L_0 = \frac{a^2c}{2\omega_p\omega_c}[(\omega_0 - \omega_c)^2 + \gamma^2]^{3/2}$$

according to (19). Because $\alpha_0'' > 0$, we are in Case 1, and since

$$1 \ll 2a[(\omega_0 - \omega_c)^2 + \gamma^2]^{1/2}$$

is satisfied by virtue of (21), condition (22a) is satisfied. On the other hand, at this distance the wave has been attenuated roughly by a factor $\exp[-\frac{1}{2}\alpha_0 L_0]$, and

$$\tfrac{1}{2}\alpha_0 L_0 = \frac{\gamma a^2}{2}\,[(\omega_0 - \omega_c)^2 + \gamma^2]^{1/2} \gg 1$$

as before. Under these conditions then, attenuation rather than broadening would seem to be the main concern in media obeying dispersion law (20).

ACKNOWLEDGMENT

The authors wish to thank Prof. L. Lewin, Prof. F. Barnes, and Prof. S. Maley for their helpful discussions and suggestions, as well as the anonymous reviewer who pointed out a serious error in the original manuscript. They also express appreciation to M. Kindgren for her preparation of the manuscript.

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